MATH 5061 Lecture 12 (Apr 7)

Important: Take-home final 4/28 2:30PM - 5/5 2:30PM

[Last Problem Set posted, due on 4/21.]

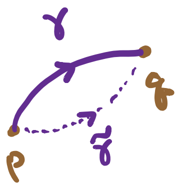
Last week: conjugate pts & minimizing properties of geodesics

- conjugate pt happens when $d(\exp_p)$ is singular at that pt.
- "completeness" of Riem. mfd.

Hopf-Rinow: geodesic completeness $\Leftrightarrow$ metric completeness

(M^n, g) $\qquad d(p,q) = \inf_{\gamma: p \to q} L(\gamma)$ $\qquad (M^n, d)$

$L(\gamma) = d(p,q)$



$\Leftrightarrow$ $\exp_p : T_pM \to M$ is well-defined on the entire T_pM.

$\Rightarrow$ Any $p, q \in M$ can be connected by a minimizing geodesic in (M^n, g).

Recall that a central question in Riem. Geometry is:

Q: Given a complete (M^n, g), how does the curvature(s) reflect the topology of M^n? (E.g.: Gauss-Bonnet)

- one example is Synge Thm: (M^{2n}, g), cpt, orientable, $K > 0 \Rightarrow \pi_1 M = 0$.
- Today: Bonnet-Myers Thm & Cartan-Hadamard Thm.

Bonnet-Myers Thm:

Let (M^n, g) be a complete Riem. manifold.

Suppose $\exists\, r > 0$ s.t. $\forall\, p \in M$, $\forall\, v \in T_pM$ where $\|v\| = 1$

$$\boxed{\mathrm{Ric}^M_p(v,v) \geq \frac{n-1}{r^2} > 0}$$

Ricci curvature of $\mathbb{S}^n(r)$

THEN, M is compact and

diam $\mathbb{S}^n(r)$

$$\sup_{p,q\in M}\{d(p,q)\} = \boxed{\mathrm{diam}(M^n, g) \leq \pi r}$$

Hence, $\pi_1(M)$ is finite.

Remark: S.Y. Cheng proved the rigidity case

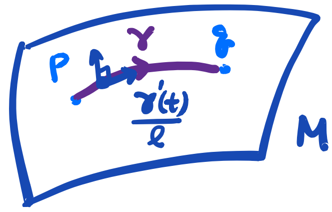
"Maximal Diameter Thm": $\mathrm{diam}\, M = \pi r \Rightarrow (M^n, g) \overset{\text{isometric}}{\cong} (\mathbb{S}^n(r), g_{\text{round}})$

Proof of Bonnet-Myers:

Idea: Use 2^{nd} variation of geodesics to establish diameter bound.

Take arbitrary points $p, q \in M$. By Hopf-Rinow, $\exists$ minimizing geodesic $\gamma : [0,1] \longrightarrow (M^n, g)$ s.t.

$\gamma(0) = p$, $\gamma(1) = q$, $L(\gamma) = d(p,q) =: \ell$



Claim: $\ell \leq \pi r$ $\quad$ ($\Rightarrow$ $\mathrm{diam}(M^n, g) \leq \pi r$)

Proof: Argue by contradiction. Suppose NOT, ie $\ell > \pi r$ $\quad(*)$

γ minimizing $\Rightarrow$ $E''(0) \geq 0$ $\forall$ variation of γ $\quad(\#)$

Fix a parallel O.N.B. $\left\{ \frac{\gamma'(t)}{\ell}, e_1(t), \dots, e_{n-1}(t) \right\}$ along γ

Define: $V_i(t) := (\sin \pi t)\, e_i(t)$ for $i = 1, \dots, n-1$

Note $V_i(0) = V_i(1) = 0 \Rightarrow$ end-pt fixing variations γ^i_s

2nd variation of energy w.r.t. γ^i_s $\Rightarrow$ $E_i''(0) = \int_0^1 \langle V_i', V_i' \rangle - \langle R(\gamma', V_i)\gamma', V_i \rangle \, dt$

$$= -\int_0^1 \langle V_i'' + R(\gamma', V_i)\gamma', V_i \rangle \, dt$$

$$= \int_0^1 \sin^2 \pi t \left[\pi^2 - \ell^2 K_{\gamma(t)}\left(\mathrm{span}\left\{ e_i(t), \tfrac{\gamma'(t)}{\ell} \right\}\right) \right] dt$$

Summing $i = 1, \dots, n-1$,

$$\sum_{i=1}^{n-1} E_i''(0) = \int_0^1 \sin^2 \pi t \left[(n-1)\pi^2 - \ell^2 \overbrace{\mathrm{Ric}^M_{\gamma(t)}\left(\tfrac{\gamma'(t)}{\ell}, \tfrac{\gamma'(t)}{\ell} \right)}^{\geq \frac{n-1}{r^2} \text{ by assumption}} \right] dt$$

$$\leq \int_0^1 \underbrace{\sin^2 \pi t}_{\geq 0} \underbrace{\left[(n-1)\pi^2 - (n-1)\frac{\ell^2}{r^2} \right]}_{< 0 \text{ by } (*)} dt < 0$$

Thus, $E_i''(0) < 0$ for SOME i, contradicts $(*)$.

By Hopf-Rinow, diam$(M, g) \leq \pi r$ + complete $\Rightarrow$ M cpt.

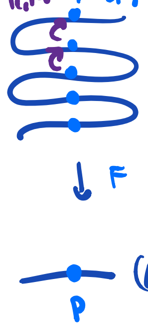
To prove $\pi_1 M$ is finite, consider its "universal cover"

$\pi_1 M \circlearrowleft \tilde{M} \leftarrow \pi_1 = 0$

deck transformation

$\tilde{M} \downarrow M$

$\pi_1 M$ $F^{-1}\{p\}$



$(\tilde{M}^n, \overbrace{F^* g}^{\tilde{g}})$ Ric $\geq \frac{n-1}{r^2}$ $\Longrightarrow$ diam$(\tilde{M}, \tilde{g}) \leq \pi r$ & $\tilde{M}$ cpt

$\downarrow F$

$\Downarrow$

p (M^n, g) Ric $\geq \frac{n-1}{r^2}$

$\pi_1 M$ finite.

$\blacksquare$

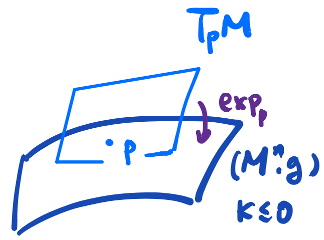
Cartan-Hadamard Thm: Let (M^n, g) be a complete Riem. mfd.

Suppose M has non-positive sectional curvature, i.e.

$$\boxed{K^M \leq 0}$$

THEN, $\exp_p : T_pM \to M$ is a covering map $\forall p \in M$.

Hence, if $\pi_1 M = 0$, then $M^n \overset{\text{diffeo}}{\cong} T_pM \cong \mathbb{R}^n$.

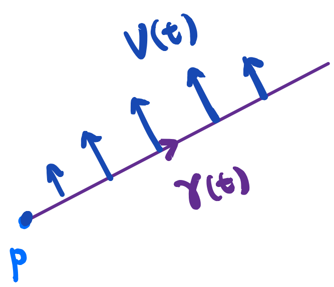


"Sketch of Proof": Idea: Jacobi field estimates.

Step 1: $\nexists$ conjugate pts on ANY geodesic in M

Since (M^n, g) is complete $\xRightarrow[\text{Rinow}]{\text{Hopf-}}$ $\exp_p : T_pM \to M$ is defined.

Let $\gamma : [0, \infty) \to (M^n, g)$ be a geodesic with $\gamma(0) = p$.



Suppose $V(t)$ is a normal Jacobi field on γ

st $V(0) = 0$, $V'(0) \neq 0$

Claim: $V(t) \neq 0 \quad \forall t \in (0, \infty)$

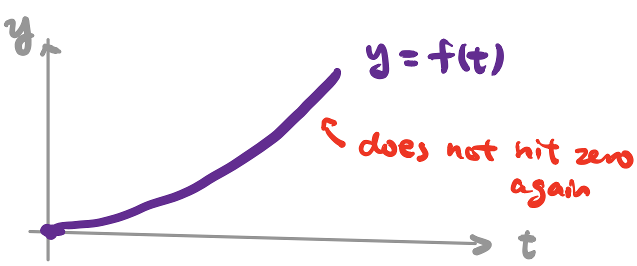
Pf: Consider the function $f(t) := \| V(t) \|^2$

$$f'(t) = 2 \langle V(t), V'(t) \rangle$$

$$f''(t) = 2 \langle V'(t), V'(t) \rangle + 2 \langle V(t), V''(t) \rangle$$

$$= 2 \langle V'(t), V'(t) \rangle - 2 \underbrace{\langle V(t), R(\gamma'(t), V(t)) \gamma'(t) \rangle}_{\leq 0 \text{ by } K \leq 0}$$

Hence, $f''(t) \geq 0 \quad \forall t \in [0, \infty)$.



$f(0) = \| V(0) \|^2 = 0$

$f'(0) = 2 \langle V(0), V'(0) \rangle = 0$

$f''(0) = 2 \| V'(0) \|^2 > 0$

Step 2: $\exp_p : T_pM \to M$ is a covering map (w.r.t some metric)

By Step 1, $d(\exp_p)_v$ is non-singular $\forall\, v \in T_pM$

$\underset{\text{IFT}}{\Rightarrow}$ $\exp_p : T_pM \to M$ is a local diffeo.

$\Rightarrow$ $\exp_p : (T_pM, \exp_p^* g) \to (M, g)$ local isometry

hence is a covering map.

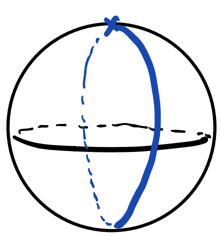
So, if M is simply-connected, since $\pi_1 T_pM = 0$, then $\exp_p$ must be a diffeomorphism.

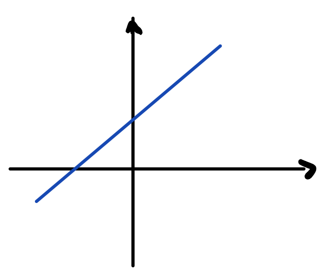
$\square$

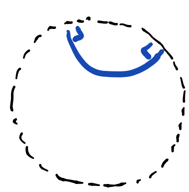
Space Forms

There are 3 "model geometries" in 2D:

$(K \equiv 1)$ Spherical	$(K \equiv 0)$ Euclidean	$(K \equiv -1)$ Hyperbolic
$(\mathbb{S}^2, g_{round})$	$(\mathbb{R}^2, g_{flat})$	$(\mathbb{H}^2, g_{hyp})$







Poincaré disk model

$\downarrow$ $\mathbb{RP}^2$ $\quad$ $\downarrow$ T^2, $\mathbb{R}\times S^1$ $\quad$ $\downarrow$ Σ_g $(g \geq 2)$

Q: Can we classify the "model geometries" in higher dimensions?

i.e. constant curvature spaces

Cartan Theorem: $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{H}^n$ are the only simply connected, complete Riem. n-manifolds with constant sectional curvatures.

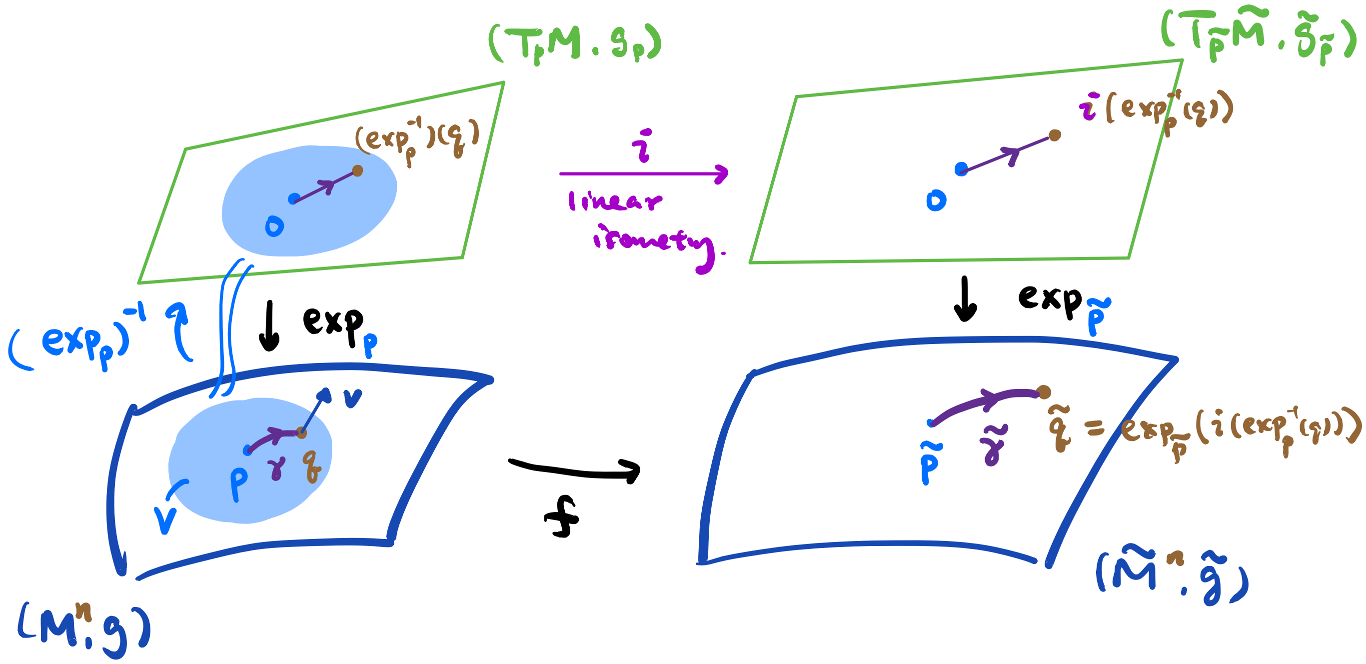
The key idea to the proof is a lemma due to Cartan - Ambrose which says that the Riem. curvature tensor R determines the Riem. metric g locally.

$$g \underset{\text{"}\int^2\text{"}}{\overset{\text{"}d^2\text{"}}{\rightleftarrows}} R$$

<u>Notation for Cartan - Ambrose Lemma</u>:

(M^n, g), $(\tilde{M}^n, \tilde{g})$ complete Riem. mfd of same dim $= n$

Fix $p \in M$, $\tilde{p} \in \tilde{M}$ and a linear isometry $i : T_pM \to T_{\tilde{p}}\tilde{M}$



Let $V \subseteq M$ be a nbd of p in M st the geodesic normal coordinate system (centered at p) is well-defined in V

<u>Define</u>: $f : V \to \tilde{M}$ by $f(q) := \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(q)$

Let $\gamma : [0, t] \to M$ be geodesic from p to q

$\tilde{\gamma} : [0, t] \to M$ be geodesic from $\tilde{p}$ to $\tilde{q}$

parallel transports along γ, $\tilde{\gamma}$

<u>Define</u>: $\phi_t : T_qM \to T_{\tilde{q}}\tilde{M}$ by $\phi_t(v) := \tilde{P}_t \circ i \circ P_t^{-1}(v)$

Cartan-Ambrose Lemma: Under the notations above:

If $\forall q \in V$, $\forall x, y, u, v \in T_qM$,

$$R(x,y,u,v) = \tilde{R}(\phi_t(x), \phi_t(y), \phi_t(u), \phi_t(v)).$$

THEN, $f: V \to f(V)$ is a local isometry.

"**Sketch of Proof**": Fix $q \in V$, and let $\gamma: [0,\ell] \to M$ p.b.a.l.

Fix $v \in T_qM$. By the choice of V, q is NOT conjugate to p

$\Rightarrow \exists$ Jacobi field $V(t)$ along $\gamma(t)$ st $V(0)=0$, $V(\ell)=v$

Choose a parallel O.N.B. $\{e_1, \dots, e_n\}$ along γ st $e_n = \gamma'$

write: $$V(t) = \sum_{j=1}^{n} \alpha_j(t)\, e_j(t)$$

Jacobi eq^{n} $\Rightarrow$ $$\alpha_j'' + \sum_{i=1}^{n} R(e_n, e_i, e_n, e_j)\, \alpha_i = 0 \quad \text{for } j=1,\dots,n$$

Define: $\tilde{V}(t) := \phi_t(V(t)) \quad \forall t \in [0,\ell]$

$\tilde{e}_j(t) := \phi_t(e_j(t))$ O.N.B. parallel along $\tilde{\gamma}$

Note: $$\tilde{V}(t) = \sum_{j=1}^{n} \alpha_j(t)\, \tilde{e}_j(t)$$

By hypothesis, $R(e_n, e_i, e_n, e_j) = \tilde{R}(\tilde{e}_n, \tilde{e}_i, \tilde{e}_n, \tilde{e}_j)$.

$\Rightarrow$ $$\alpha_j'' + \sum_{i=1}^{n} \tilde{R}(\tilde{e}_n, \tilde{e}_i, \tilde{e}_n, \tilde{e}_j)\, \alpha_i = 0 \quad \text{for } j=1,\dots,n$$

So, $\tilde{V}(t)$ is a Jacobi field along $\tilde{\gamma}$ with $\tilde{V}(0)=0$.

Since $P_t, \tilde{P}_t$ are isometries, we have $\|\tilde{V}(\ell)\| = \|V(\ell)\|$

Finally, one checks that $\tilde{V}(\ell) = df_q(v)$, i.e. df_q is isometry.
(Ex.)

$\square$